# 16.1 Vector Fields.

<u>Def. 1.</u> $D \subseteq \mathbb{R}^2$

$\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow$ Vectors in $\mathbb{R}^2$.

$(x, y) \longrightarrow \vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

Where $P, Q$ scalar functions of two variables:

$P : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ , $Q : D \subseteq \mathbb{R}^2 \to \mathbb{R}$.

Such $\vec{F}$ is called a vector field on $\mathbb{R}^2$.

<u>Examples:</u>

a) $\vec{F}(x,y) = \langle x, y \rangle$

$\longrightarrow$ b) $\vec{F}(x,y) = \langle 1, x \rangle$

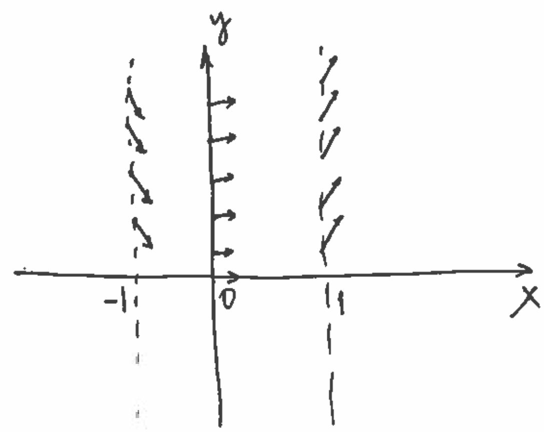
c) $\vec{F}(x,y) = \langle x-2, x+1 \rangle$

Ask Students to obtain by hand Some Vectors

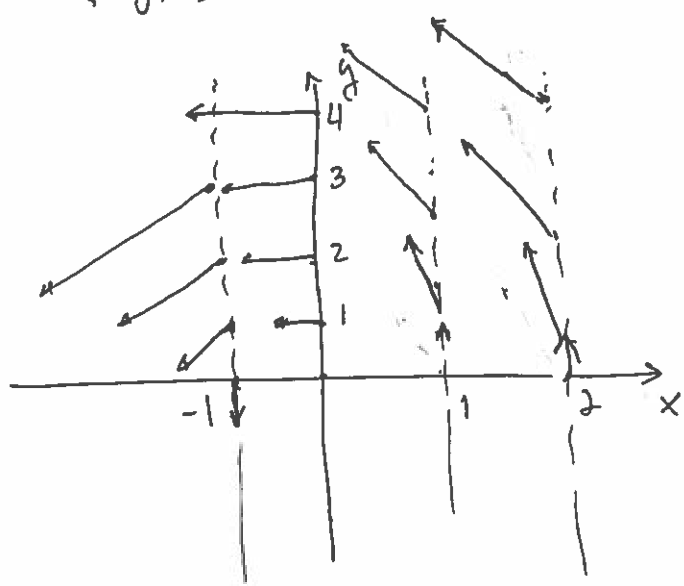
Show Maple worksheet.

$\longrightarrow$ d) $\vec{F}(x,y) = \langle -y, x \rangle$.

b) $\vec{F}(x,y) = \langle 1, x \rangle$.




d) $\vec{F}(x,y) = \langle -y, x \rangle$.



Def. 2

$D \subseteq \mathbb{R}^3$

$\vec{F}: D \subseteq \mathbb{R}^3 \longrightarrow$ Vectors in $\mathbb{R}^3$

$(x,y,z) \longrightarrow \vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

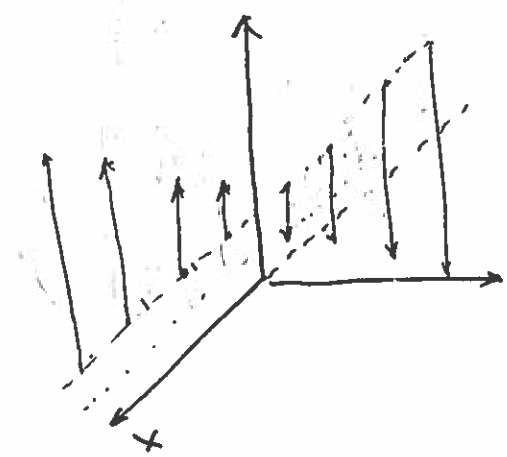
$P, Q, R : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$

Scalar functions of three Variables.

$\vec{F}$ is called a vector field on $\mathbb{R}^3$.

Exercise 16.1 #9

$\vec{F}(x,y,z) = x\,\hat{k}$

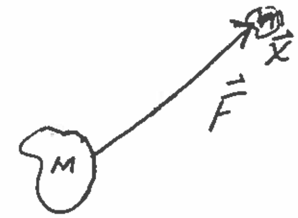



Show Maple worksheet.

# Newton's Law of Gravitation

## Gravitational Field:

$$\vec{F}(\vec{x}) = \frac{-mMG}{|\vec{x}|^2} \frac{\vec{x}}{|\vec{x}|}$$




$\vec{x}$: Position vector of object with mass

## Gradient Fields:

Def.- Given $f(x,y,z)$ scalar fn. of 3 Variables, with first partial derivatives, the function

$$\nabla f : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \longrightarrow \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

is called a gradient field corresponding to $f$.

## Conservative Vector Fields:

Def.- A vector field $\vec{F}(\vec{x})$ is called conservative vector field if it is the gradient of some scalar function $f$.

i.e., $\vec{F}(\vec{x}) = \nabla f(\vec{x})$

$f$ is said to be "a potential function for $\vec{F}$."

Consider the vector fields

a) $\vec{F}(x,y) = \langle x-y,\ y-x \rangle$

b) $\vec{F}(x,y,z) = \langle z\cos y,\ xz\sin y,\ x\cos y \rangle$

are they conservative?

answ: a) yes, since

$f(x,y) = \frac{1}{2}(x-y)^2$ is such that

$\nabla f(x,y) = \langle x-y,\ y-x \rangle$

b) No. we'll prove it later in Sect. 16.5.

The Gravitational field is a conservative vector field.

Notice that

If $r(x,y,z) = (x^2+y^2+z^2)^{1/2} = |\vec{x}|$

then, $r_x(x,y,z) = \frac{\partial r}{\partial x}(x,y,z) = \frac{\cancel{2}x}{\cancel{2}(x^2+y^2+z^2)^{1/2}} = \frac{x}{(x^2+y^2+z^2)^{1/2}}$

$$= \boxed{\frac{x}{r(x,y,z)}}$$

Similarly,

$$\boxed{r_y(x,y,z) = \frac{y}{(x^2+y^2+z^2)^{1/2}} = \frac{y}{r(x,y,z)}}, \quad \boxed{r_z(x,y,z) = \frac{z}{(x^2+y^2+z^2)^{1/2}} = \frac{z}{r(x,y,z)}}$$

The Gravitational field is

$$\boxed{\vec{F}(\vec{x}) = \vec{F}(x,y,z) = \frac{-mMG}{|\vec{x}|^2}\frac{\vec{x}}{|\vec{x}|}}$$

or the force acting on a mass $m$ located at point $P(x,y,z)$ due to a mass $M$ located at the origin.

More precisely,

$$\boxed{\vec{F}(\vec{x}) = -mMG\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle}$$

We will show that $\vec{F}$ is a conservative vector field

Another representation:

$$\boxed{F(\vec{x}) = \frac{-mMG}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle}$$

In fact,

Consider $$\boxed{f(x,y,z) = \frac{mMG}{(x^2+y^2+z^2)^{1/2}} = \frac{mMG}{r(x,y,z)}}$$

Then,

$$\nabla f(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \rangle$$

Applying chain rule:

$$\nabla f(x,y,z) = \langle f_r r_x,\ f_r r_y,\ f_r r_z \rangle,$$

where $$f_r(\mathbf{r}(x,y,z)) = \frac{-mMG}{r^2(x,y,z)}$$

Therefore,

$$\nabla f(x,y,z) = -mMG \left\langle \frac{r_x}{r^2},\ \frac{r_y}{r^2},\ \frac{r_z}{r^2} \right\rangle$$

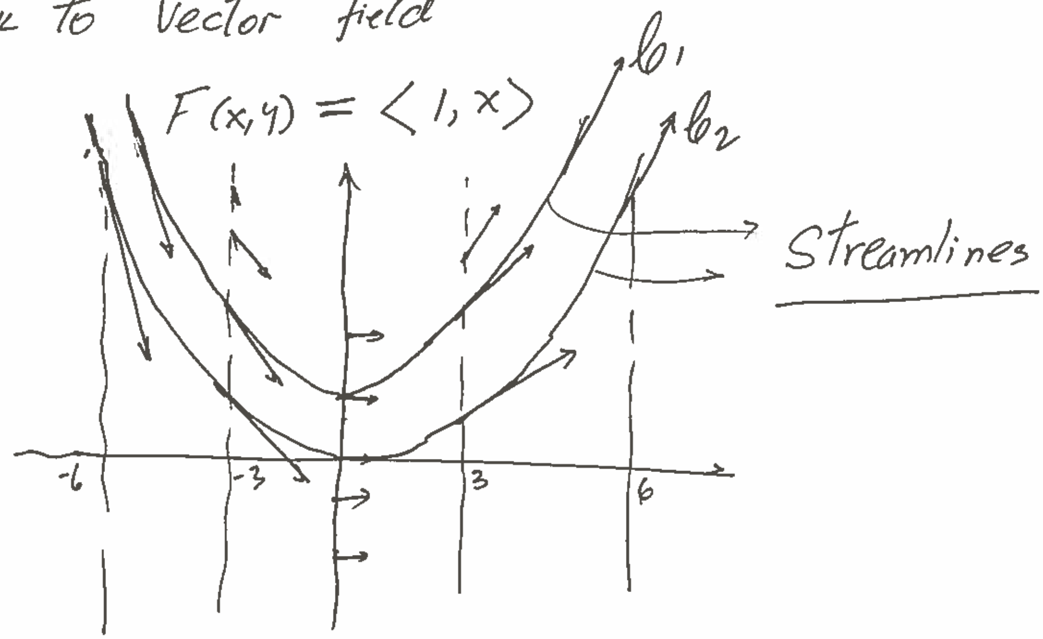
$$= -\frac{mMG}{r^2} \langle r_x, r_y, r_z \rangle = \frac{-mMG}{r^2} \left\langle \frac{x}{r},\ \frac{y}{r},\ \frac{z}{r} \right\rangle$$

or $$\nabla f(x,y,z) = \frac{-mMG}{r^2} \frac{1}{r} \langle x,y,z \rangle = -\frac{mMG}{r^2} \frac{\vec{x}}{r}$$

or $$\nabla f(x,y,z) = \frac{-mMG}{|\vec{x}|^2} \frac{\vec{x}}{|\vec{x}|} = \vec{F}(\vec{x})$$

The scalar function $f(x,y,z) = \dfrac{mMG}{(x^2+y^2+z^2)^{1/2}} = \dfrac{mMG}{r}$ is a potential function for the gravitational field.

Back to vector field

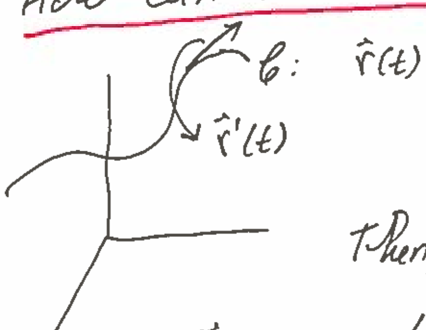



Def: (Streamlines or flow lines)

The streamlines or flow lines of a vector field are the paths followed by a particle whose velocity field is the given vector field.

It implies that the vectors in a vector field are tangent to the flow lines.

How can we obtain a math expression for the streamlines from a vector field ?




Consider a curve

$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$.

Then, tangent vector to $C$ at $(x,y,z)$ is

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle .$$

Now, if we know $\vec{F}(x,y,z)$ Vector field, then we can integrate its components to find and expression for $\mathcal{C}$.

<u>Example</u> (Exercise 36 (16.1))

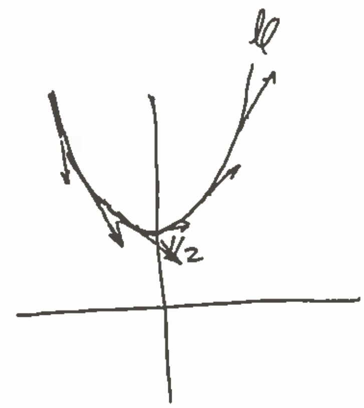
Find flow lines for $\boxed{\vec{F}(x,y) = \langle 1, x\rangle}$.

<u>answ:</u>

First, we consider the parametric form of streamlines $\mathcal{C}: \vec{r}(t) = \langle x(t), y(t)\rangle$

then $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle = \vec{F}(x(t), y(t)) = \langle 1, x(t)\rangle$

Therefore,
$$\boxed{\begin{aligned} x'(t) &= 1 \quad &(1) \\ y'(t) &= x(t) \quad &(2) \end{aligned}}$$




Solving (1) and Subst. into (2)

$\boxed{x(t) = t + c}$ and $y'(t) = t + c$

implies $\boxed{y(t) = \frac{t^2}{2} + ct + d}$

If we were interested in the curve passing through $(0,1)$ we consider IC's $\boxed{x(0) = 0, \quad y(0) = 1.}$

$0 = 0 + c \Rightarrow c = 0, \qquad 1 = 0 + d \Rightarrow d = 1$

or $\boxed{\vec{r}(t) = \langle t, \frac{t^2}{2} + 1\rangle}$. or $\boxed{y = \frac{x^2}{2} + 1}$ If $x = t$